Dependence of Faraday rotation and magneto-optical figure of merit on dielectric tensor elements in materials with uniaxial symmetry

Alexander R. Taussig, Gerald F. Dionne,* and C. A. Ross†
Department of Materials Science and Engineering, Massachusetts Institute of Technology (MIT), 77 Massachusetts Avenue, Cambridge, Massachusetts 02139, USA
(Received 11 June 2007; published 24 January 2008)

We derive the functional dependence of the specific Faraday rotation Θ, optical absorption α, and magneto-optical figure of merit $F \equiv |\Theta|/\alpha$ on the dielectric tensor elements of a uniaxial, magneto-optically active material in a wavelength regime of relative transparency. In addition, we calculate F as a function of $\omega < 2.2$ eV for the diamagnetic transition of the octahedrally coordinated Fe^{3+} in bismuth doped yttrium iron garnet ($\omega_0 = 3.15$ eV) and show that F achieves a local maximum value in this frequency regime at $\omega = 1.25$ eV. We also discuss the implications of this result in rare-earth iron garnets for bulk magneto-optical isolators and in orthoferrites for thin film devices. Finally, we discuss the importance of controlling linear birefringence in thin film isolators and its impact on the usefulness of F.

DOI: 10.1103/PhysRevB.77.012407 PACS number(s): 78.20.Ls, 78.30.−j

I. INTRODUCTION

Over the next ten years, it is predicted that conventional silicon-based electronics will approach a fundamental bandwidth limit caused by physical loss mechanisms in electronic transmission media.[1] A solution to this problem may eventually be provided by the field of microphotonics, which promises to extend Moore's law by utilizing low-loss waveguides to propagate optical signals in a chip fabricated on a conventional silicon substrate. Ideally, the other components that are required to make an optical link (e.g., a laser source, filter, modulator, and detector) could all be integrated seamlessly as well, resulting in a processing unit with orders of magnitude more bandwidth and power efficiency than those currently available on the market.

An essential component of such a photonic architecture is the isolator, a one-way valve for light. It protects a laser light source coupled into an optical circuit from back reflections that can result in significant injection noise, which can lower the maximum operational speed of the link and reduce the lifetime of the laser source.[2] A magneto-optical isolator employs the Faraday effect or Faraday rotation, which arises due to quantum transitions between orbital angular momentum states of electric dipoles magnetically aligned with the direction of propagation.[3] This results in the rotation of the electric vector of a linearly polarized beam of light as it passes through an active medium.

Typical magneto-optical isolator materials at near infrared telecommunications wavelengths (e.g., 1550 nm) are based on rare-earth iron garnets [$R_3Fe_5O_{12}$, where R is a rare-earth cation (Ref. 4, p. 163)]. Orthoferrites ($AFeO_3$, where A is a trivalent cation[5]) have been used in the past for magneto-optical data storage, but also show great promise for use in novel thin film isolators, which can be grown epitaxially on semiconductor substrates and semiconductor-compatible oxides. Both materials have tetrahedrally and/or octahedrally coordinated Fe^{3+} cations, which are the main source of Faraday rotation at infrared wavelengths. Hence, both can possess a relatively large specific Faraday rotation Θ [100–1000°/cm (Refs. 6 and 7)] and a relatively low near infrared absorption α [0.03–0.5 cm^{-1} (Refs. 8 and 9)], giving each of them a high magneto-optical figure of merit $F \equiv |\Theta|/\alpha$.

Despite the usefulness of F as a material selection criterion, there has been little thought given to it on a theoretical basis. Although many papers report values of F for different magneto-optical materials (e.g., 9.1°/dB for yttrium iron garnet and 25.8°/dB for the same material doped with ytterbium and bismuth[10]), a concise derivation of its dependence on the dielectric tensor elements of a material has not been presented previously. In this report, we therefore derive Θ, α, and F each as a function of these dielectric tensor elements. Such a relationship could enable F to be maximized with respect to operating wavelength or chemical composition, in particular. More details on the intermediate steps of this derivation can be found in Ref. 11.

II. FARADAY ROTATION

A. General assumptions

Certain simplifying assumptions can be made in the case of many transparent materials. We begin with Maxwell's equations for the electric field $\boldsymbol{E}$, electric displacement $\boldsymbol{H}$, magnetic flux density $\boldsymbol{B}$, and magnetic field $\boldsymbol{H}$, and assume no free charge in the material, i.e., $\rho_f = 0$. Additionally, we assume that the material is a linear dielectric (i.e., $\boldsymbol{D} = \boldsymbol{\epsilon}\boldsymbol{E}$, $\boldsymbol{B} = \mu\boldsymbol{H}$) and that Ohm's law is true ($\boldsymbol{J}_f = \sigma\boldsymbol{E}$), where $\boldsymbol{\epsilon}$, μ, and σ are all symmetric, second-rank tensors. Also, we assume that the material is finite in the z direction but extends infinitely in the x-y plane (i.e., the thin film approximation) and has a crystal structure with at least uniaxial symmetry about the z axis. Moreover, we assume that the material is insulating, so that terms proportional to σ are negligible. μ may be trivially set equal to the identity tensor multiplied by μ_0, the permeability of free space, in the case of uniaxial symmetry, according to Ref. 12. Finally, if ϵ_0 is the permittivity of free space, we assume that the dielectric tensor $\boldsymbol{\epsilon}$ has the form

$$\boldsymbol{\epsilon} = \epsilon_0 \begin{pmatrix} \epsilon_d & -ig & 0 \\ ig & \epsilon_d & 0 \\ 0 & 0 & \epsilon_z \end{pmatrix}, \tag{1}$$

which is its most general form in configurations with uniaxial symmetry, as demonstrated in Ref. 12.

B. Boundary conditions and normal modes

With the above assumptions, Maxwell's equations reduce to the familiar wave equation for $\boldsymbol{E}$,

$$\nabla^2 \boldsymbol{E} - \mu_0 \boldsymbol{\epsilon} \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0. \tag{2}$$

We assume the ansatz of a plane wave of magnitude $\boldsymbol{E}_0$ propagating in the z direction with angular frequency ω, i.e., $\boldsymbol{E}=\boldsymbol{E}_0 \exp\{-i\omega(t-nz/c)\}$. n is the complex index of refraction of the material and will be the eigenvalue when this ansatz is substituted into Eq. (2). Doing so reveals the eigenvalues $n_\pm^2=\epsilon_d \pm g$ and the orthonormal eigenmodes $\boldsymbol{C}_\pm=(1/\sqrt{2})(\hat{x}\pm i\hat{y})\exp\{-i\omega(t-n_\pm z/c)\}$, which represent right- and left-handed circularly polarized light, respectively.[21] Because these are the eigenmodes, any $\boldsymbol{E}$ field in the material may be represented as a linear combination of $\boldsymbol{C}_+$ and $\boldsymbol{C}_-$. The boundary condition that $\boldsymbol{E}$ is linearly polarized in the x direction when it enters the material at $z=0$ implies that the equation for $\boldsymbol{E}$ is

$$\boldsymbol{E}(z,t) = E_0 \begin{pmatrix} \cos(\omega \Delta n z/c) \\ -\sin(\omega \Delta n z/c) \end{pmatrix} e^{-i\omega(t-n_0 z/c)}, \tag{3}$$

where $\Delta n \equiv (n_+ - n_-)/2$ and $n_0 \equiv (n_+ + n_-)/2$ are defined thus for convenience.

C. Transparent approximation

When Δn in Eq. (3) is real, the Faraday rotation angle between $\boldsymbol{E}$ and the x axis is simply $\theta=\tan^{-1}(E_y/E_x)$. The specific Faraday rotation $\Theta \equiv \theta/z$ is then $-\omega\Delta n/c$. When Δn and n_0 are complex, however, the equation for Θ becomes far more complicated. To simplify matters, we make the so-called "transparent approximation,"

$$|\epsilon_d'| \gg \max(|\epsilon_d''|, |g'|, |g''|), \tag{4}$$

where the above variables are defined as real quantities in the relations $\epsilon_d=\epsilon_d'+i\epsilon_d''$ and $g=g'+ig''$. Essentially, this approximation implies that the material is weakly absorbing and that magneto-optic effects are small compared to the normal dielectric response of the material. As long as ω is far from an electronic transition that causes Faraday rotation (see Appendix), as is the case in the near infrared with iron garnets or orthoferrites (Ref. 4, p. 35), this approximation is valid.

D. Derivation

We seek to solve for the components of $\boldsymbol{E}(z,t)$ for complex Δn and n_0, from which we will calculate the specific Faraday rotation Θ. It is useful, therefore, to solve for Δn and n_0 in terms of the real and imaginary parts of the dielectric tensor elements. It is easy to show that

$$n_0 = \frac{\sqrt{\epsilon_d}}{2}(\sqrt{1+Q} + \sqrt{1-Q}), \tag{5}$$

$$\Delta n = \frac{\sqrt{\epsilon_d}}{2}(\sqrt{1+Q} - \sqrt{1-Q}), \tag{6}$$

where $Q \equiv g/\epsilon_d \ll 1$ according to the transparent approximation in Eq. (4). Using a Taylor series expansion to first order in Q, we find that $n_0 \approx \sqrt{\epsilon_d}$ and $\Delta n \approx (\sqrt{\epsilon_d}/2)Q$. With some algebraic manipulation and repeated use of the Taylor series to leading order in b/ϵ_d'—where b can be ϵ_d'', g', or g''—it can be shown that

$$n_0 = n_0' + in_0'' = \sqrt{\epsilon_d'} + i\frac{\epsilon_d''}{2\sqrt{\epsilon_d'}}, \tag{7}$$

$$\Delta n = \Delta n' + i\Delta n'' = \frac{g'}{2\sqrt{\epsilon_d'}} + i\frac{g''}{2\sqrt{\epsilon_d'}}. \tag{8}$$

$\boldsymbol{E}$ can be expressed with these complex coefficients as $\boldsymbol{E}=E_0\boldsymbol{v}\exp\{-i\omega t + i\zeta(n_0'+in_0'')\}$, where $\zeta \equiv \omega z/c$ and

$$\boldsymbol{v} \equiv \begin{pmatrix} \cos\zeta\Delta n' \cosh\zeta\Delta n'' - i\sin\zeta\Delta n' \sinh\zeta\Delta n'' \\ -\sin\zeta\Delta n' \cosh\zeta\Delta n'' - i\cos\zeta\Delta n' \sinh\zeta\Delta n'' \end{pmatrix}. \tag{9}$$

Using the notation of and the relationships derived in (Ref. 4, p. 29), $\chi \equiv E_y/E_x = v_y/v_x$, so that the Faraday rotation angle θ may be written as

$$\tan 2\theta = \frac{2}{1-|\chi|^2}\,\mathrm{Re}\{\chi\}. \tag{10}$$

After substituting the components of $\boldsymbol{v}$ given in Eq. (9) into the definition of χ and substituting χ itself into Eq. (10), we arrive at the result $\theta=-\zeta\Delta n'$, from which we can quickly recognize that

$$\Theta \equiv \frac{\theta}{z} = -\frac{\omega}{2c}\frac{g'}{\sqrt{\epsilon_d'}}. \tag{11}$$

III. ABSORPTION

In general, the two normal modes of propagation in the material $\boldsymbol{C}_\pm$ will have two different absorption coefficients $\alpha_\pm$. Because $\alpha_\pm z$ is defined as *twice* the overall exponential prefactor in the equation for $\boldsymbol{E}$ (since $\alpha_\pm$ refers to the decay of the intensity of the light, which is proportional to $|\boldsymbol{E}|^2$), we find $\alpha_\pm$ by rearranging terms in the equation for $\boldsymbol{E}$ and doubling the evanescent exponent. Writing $\boldsymbol{E}$ as a linear combination of its normal modes, we have

$$\boldsymbol{E}(z,t) = (E_0/\sqrt{2})(\boldsymbol{C}_+ + \boldsymbol{C}_-) = \frac{E_0}{2\sqrt{2}} e^{-\zeta(n_0''+\Delta n'')} \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-i\omega t + i\zeta(n_0'+\Delta n')}$$

$$+ \frac{E_0}{2\sqrt{2}} e^{-\zeta(n_0''-\Delta n'')} \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-i\omega t + i\zeta(n_0'-\Delta n')}. \quad (12)$$

In Eq. (12), the first term on the right-hand side corresponds to the right (+) mode, while the second corresponds to the left (−) mode. Thus, by inspection, $\alpha_\pm z/2 = \zeta(n_0'' \pm \Delta n'')$, or

$$\alpha_\pm = \frac{\omega}{c}\sqrt{\epsilon_n'}\left(\frac{\epsilon_d''}{\epsilon_d'} \pm \frac{g''}{\epsilon_d'}\right). \quad (13)$$

IV. FIGURE OF MERIT

The magneto-optical figure of merit is $F \equiv |\Theta|/\alpha_m$, where $\alpha_m = \max(\alpha_\pm)$. The reason we take the maximum of $\alpha_\pm$ to be the characteristic absorption is both to make our figure of merit conservative and to recognize that if the two differently attenuated normal modes, which are transmitted through the material, are added back together at a polarizer, the more attenuated mode (i.e., α_m) will limit the amplitude of any linearly polarized light that can be constructed by adding them back together. According to Eq. (13), if we assume $g'' > 0$, $\alpha_m = \alpha_+ > \alpha_-$, so, using Eqs. (11) and (13) for Θ and α_+, respectively, we can write

$$F = \frac{|\Theta|}{\alpha_+} = \frac{|g'|}{2(\epsilon_d'' + g'')}. \quad (14)$$

It should be noted that, although Eq. (14) seems to show a dependence of F on three independent variables, this is not the case. Because the Kramers-Kronig dispersion relations allow us to numerically calculate g' from g'', if we know the latter as a function of frequency ω, or vica versa, Eq. (14) really only contains two independent variables.

V. DISCUSSION

The result in Eq. (14) is both intuitive and useful. It provides us with confirmation that in order to make a good magneto-optical isolator (or other device based on Faraday rotation), one must pick a material which, at a given wavelength, has a large value of g' and small values of ϵ_d'' and g''. Furthermore, within g are encoded the quantum mechanical origins of Faraday rotation in the material, described in detail in the Appendix. g is tunable by choice of operating wavelength and compositional variation (e.g., bismuth substitution into yttrium iron garnet[13]).

To illustrate the concept of maximizing F with respect to wavelength, or the equivalent optical frequency ω (given here in units of eV, implicitly multiplied by $\hbar$), we show in Fig. 1 the results of a calculation of $g'(\omega)$, $g''(\omega)$, and $F(\omega)$ for the diamagnetic electric dipole transition of the octahedral Fe^{3+} in the garnet $Y_{2.75}Bi_{0.25}Fe_5O_{12}$, as described in Ref. 14. In our analysis, we make the simplifying assumption that $\epsilon_d'' \ll g''$ in order to more clearly illustrate the behavior of F, and, as in Ref. 14, we assume that $\epsilon_d' \approx 2.3^2 = 5.29$. We define the region of the transparent approximation to be that in which both g'/ϵ_d' and g''/ϵ_d' are less than 1/25; this corresponds to values of g' and g'' less than 0.212, or $\omega < 2.20$ eV, which is the region shown in Fig. 1. A maximum in F of 1665°/dB occurs within the transparent region at $\omega = 1.25$ eV (992 nm), and at $\omega = 0.80$ eV (1550 nm), $F = 1434°/\text{dB}$.

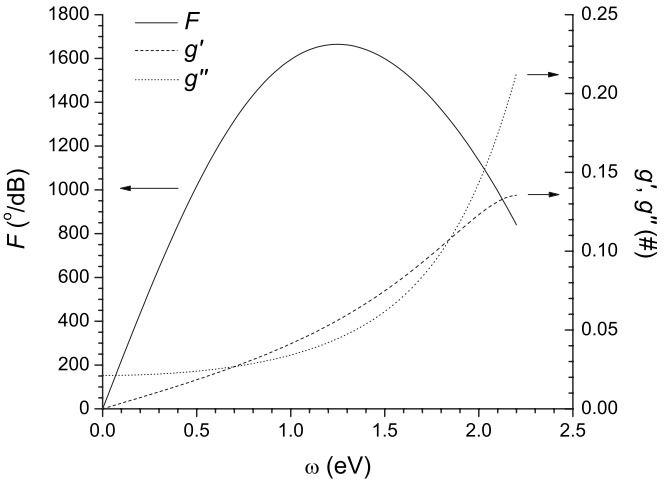

FIG. 1. F, g', and g'' each as a function of ω for the octahedral Fe^{3+} diamagnetic electric dipole transition in $Y_{2.75}Bi_{0.25}Fe_5O_{12}$. This transition exists at $\omega_0 = 3.15$ eV, with $\Gamma = 0.54$ eV, $\Delta = 0.27$ eV, and $\omega_p^2 f = 8$ eV^2 (Ref. 14). Shown here is the energy regime for which g'/ϵ_d' and g''/ϵ_d' are both less than 1/25, so that the transparent approximation holds true throughout.

These values of F are unrealistically high because we have only considered a single transition; in garnets, tetrahedral Fe^{3+} transitions execute Faraday rotation in the opposite direction, leading to values of F 2 orders of magnitude lower. Orthoferrites, however, have only this single octahedral transition; so, given the same strong internal magnetic field, significantly higher figures of merit should be achievable. Although bismuth orthoferrite, for instance, is an antiferromagnet with a low net moment due to spin canting,[15] the partial substitution of various transition metal cations on the Fe site may induce a large enough internal field to lead to significant values of F.[11,16]

If we relax the assumption of a uniaxial material and allow the two nontrivial diagonal components to differ (i.e., $\epsilon_{xx} \neq \epsilon_{yy}$), however, the material quickly becomes birefringent and F becomes less useful by itself. As discussed in Ref. 17 and, more recently, in Ref. 18, linearly birefringent materials such as orthoferrites, despite a large *intrinsic* Θ, only admit a few degrees of Faraday rotation for any length of material. Unlike garnets, which have a cubic structure in bulk, the deviation of most orthoferrites from uniaxial symmetry at room temperature leads to significant birefringence and therefore limits the Faraday rotation that can be achieved in these materials. The bulk birefringence of orthoferrites may be modified in thin films by making use of shape or strain effects, as has been demonstrated in garnet films,[19] and such techniques may enable them to be used in thin film magneto-optical devices.

ACKNOWLEDGMENTS

We wish to acknowledge the helpful contributions of Lei Bi and Dr. Hyun-Suk Kim, as well as the encouragement of MIT Lincoln Laboratory. This research was funded by the National Science Foundation.

APPENDIX

The permittivity tensor elements of Eq. (1) are derived from electric dipole transitions between an orbital angular momentum singlet S term ($L_z=0$) and a triplet P term ($L_z=0, \pm 1$), subject to the selection rule $\Delta L_z = \pm 1$.[3] For the effects to be cooperative, the orbital magnetic moments must be aligned with the z axis. In the Faraday rotation case presented in Fig. 1, P is the excited state that is split by 2Δ, creating separate transitions at $\omega_{0\pm}=\omega_0 \pm \Delta$. Since the amplitudes of the two circular polarization modes designated as $\boldsymbol{C}_\pm$ vary spatially according to $(1/\sqrt{2})(x \pm iy)$, a relation for the off-diagonal element g can be obtained from the subtraction of the two Lorentzian-shaped resonance lines of half-width Γ and magnitude characterized by the product of the squared plasma frequency $\omega_p^2=Ne^2/m_e$ and the quantum oscillator strengths $f_\pm = m_e\omega_{0\pm}|\langle z|L_x \pm iL_y|(1/\sqrt{2})(x \pm iy)\rangle|^2$. In the present example,[13,20]

$$g' + ig'' \approx \omega_p^2 f\left(\frac{\Delta}{2\omega_0}\right)\left\{\frac{(\omega_0-\omega)^2-\Gamma^2+2i(\omega_0-\omega)\Gamma}{[(\omega_0-\omega)^2+\Gamma^2]^2}\right\}, \tag{A.1}$$

where $f \approx 2f_\pm$ and $\Delta \ll \Gamma$. A corresponding relation for the diagonal element is expressed as

$$\epsilon_d' + i\epsilon_d'' \approx 1 + \omega_p^2 \frac{f}{2}\left\{\frac{\omega_0^2-\omega^2+\Gamma^2+2i\omega\Gamma}{(\omega_0^2-\omega^2+\Gamma^2)^2+4\omega^2\Gamma^2}\right\}. \tag{A.2}$$

*Also at Lincoln Laboratory, MIT.
†caross@mit.edu

[1] J. Bautista, M. Morse, and J. Swift, 2005 Communications Technology Roadmap, MIT Microphotonics Center.
[2] B. M. Holmes, D. C. Hutchings, and J. J. Bregenzer, Mater. Res. Soc. Symp. Proc. **834**, J4.4.1 (2005).
[3] G. F. Dionne, J. Appl. Phys. **97**, 10F103 (2005).
[4] A. K. Zvezdin and V. A. Kotov, *Modern Magnetooptics and Magnetooptical Materials* (Institute of Physics, London, UK, 1997).
[5] A. S. Bhalla, R. Guo, and R. Roy, Mater. Res. Innovations **4**, 3 (2000).
[6] G. B. Scott and D. E. Lacklison, IEEE Trans. Magn. **12**, 292 (1976).
[7] W. J. Tabor, A. W. Anderson, and L. G. Van Uitert, J. Appl. Phys. **41**, 3018 (1970).
[8] D. L. Wood and J. P. Remeika, J. Appl. Phys. **38**, 1038 (1967).
[9] D. L. Wood, J. P. Remeika, and E. D. Kolb, J. Appl. Phys. **41**, 5315 (1970).
[10] W. Zhao, Sens. Actuators, A **A89**, 250 (2001).
[11] A. R. Taussig, Master's thesis, Massachusetts Institute of Technology, 2007.
[12] P. S. Pershan, J. Appl. Phys. **38**, 1482 (1967).
[13] S. Wittekoek, T. J. A. Popma, J. M. Robertson, and P. F. Bongers, Phys. Rev. B **12**, 2777 (1975).
[14] G. F. Dionne and G. A. Allen, J. Appl. Phys. **73**, 6127 (1993).
[15] C. Ederer and N. A. Spaldin, Phys. Rev. B **71**, 060401 (2005).
[16] G. F. Dionne, A. R. Taussig, M. Bolduc, L. Bi, and C. A. Ross, J. Appl. Phys. **101**, 09C525 (2007).
[17] W. J. Tabor and F. S. Chen, J. Appl. Phys. **40**, 2760 (1969).
[18] S. R. Woodford, A. Bringer, and S. Blügel, J. Appl. Phys. **101**, 053912 (2007).
[19] H. Dötsch, N. Bahlmann, O. Zhuromskyy, M. Hammer, L. Wilkens, R. Gerhardt, and P. Hertel, J. Opt. Soc. Am. B **22**, 240 (2005).
[20] G. F. Dionne and G. A. Allen, J. Appl. Phys. **73**, 6130 (1993).
[21] Note that we have dropped the z component of $\boldsymbol{E}$ because it is identically zero for a plane wave propagating in the z direction in this type of medium.